1. (10 points) Prove that the determinant of a Householder reflector is negative one.

2. (15 points) Let $\mathbf{x}_j \in \mathcal{R}^m$ be the j-th column of $X \in \mathcal{R}^{m\times n}$. Let $\mathbf{y} \in \mathcal{R}^m$ and $\lambda > 0$ be given. Given a vector $\mathbf{w} \in \mathcal{R}^n$, define the following function

$$J(\mathbf{w}) = \|X\mathbf{w} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_1.$$

Letting the i-th component w_i of $\mathbf{w}$ vary and the other components of $\mathbf{w}$ be fixed, consider the following one-variable minimization problem reduced from $J(\mathbf{w})$:

$$\begin{aligned}
\min_{w_i} f(w_i) &\equiv \min_{w_i} \|\sum_{j=1}^{n} w_j\mathbf{x}_j - \mathbf{y}\|_2^2 + \lambda|w_i| + \lambda\sum_{j\neq i}|w_j| \\
&= \min_{w_i} \|w_i\mathbf{x}_i + \mathbf{r}\|_2^2 + \lambda|w_i| + C \\
&= \min_{w_i} \sum_{j=1}^{m}(w_i x_{ji} + r_j)^2 + \lambda|w_i| + C,
\end{aligned}$$

where $\mathbf{r} \equiv \sum_{j\neq i} w_j\mathbf{x}_j - \mathbf{y}$ is in $\mathcal{R}^m$ with $\mathbf{r} = (r_k)_{m\times 1}$, and $C = \lambda\sum_{j\neq i}|w_j|$. Show that the optimal solution w_i^* for the above minimization problem is given by

$$w_i^* = \begin{cases} 0 & \text{if} \quad |a| \leq \lambda, \\ \frac{-\lambda+a}{b} & \text{if} \quad \frac{-\lambda+a}{b} > 0, \\ \frac{\lambda+a}{b} & \text{if} \quad \frac{\lambda+a}{b} < 0, \end{cases}$$

where $a = -\sum_{j=1}^{m} 2x_{ji}r_j$ and $b = \sum_{j=1}^{m} 2x_{ji}^2$.

3. (10 points) Let $A \in C^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

4. (15 points) Let $\varepsilon > 0$ be given, $k << \min(m, n)$, $A \in R^{m\times n}$, $C \in R^{m\times k}$, and $B \in R^{k\times n}$. Assume that

$$\|A - CB\| \leq \epsilon,$$

where $\|\cdot\|$ denotes the matrix 2-norm, and B and C have rank k. Further suppose that A is not available, and only B and C are available. **Without forming** the product of C and B, design an efficient algorithm to compute an approximate reduced QR of A so that the following holds,

$$\|A - QR\| \leq \varepsilon,$$

where Q is an orthonormal matrix and R is upper triangular.

5. Let $A \in \mathcal{R}^{m\times n}$, $\operatorname{rank}(A) = r$, and $\mathbf{b} \in \mathcal{R}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{R}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{R}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{R}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v}\in\mathcal{R}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (10 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.

6. Let $A \in R^{n\times n}$ be symmetric and positive definite and $n = j + k$. Partition A into the following 2 by 2 blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is $j \times j$ and A_{22} is $k \times k$. Let R_{11} be the Cholesky factor of A_{11}: $A_{11} = R_{11}^T R_{11}$, where R_{11} is upper triangular with positive main-diagonal entries. Let $R_{12} = \left(R_{11}^{-1}\right)^T A_{12}$ and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$.

   (a) (5 points) Prove that A_{11} is positive definite.

   (b) (5 points) Prove that
   $$\tilde{A}_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

   (c) (5 points) Prove that $\tilde{A}_{22}$ is positive definite.

7. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (5 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (5 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (5 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.